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On Best Simultaneous Approximation in Normed Linear Spaces

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In this paper we consider the problem of simultaneous approximation of a subset F of a Banach space B by elements of another subset $S \subset B$. Results are obtained on the existence, uniqueness, and characterization of best simultaneous approximations.

1. INTRODUCTION

Several authors have studied the problem of simultaneous approximation. Dunham [8], Diaz and McLaughlin [6, 7], Ling *et al.* [13, 14] considered simultaneous Chebyshev approximation of two real-valued functions defined on the interval [0, 1]. The problem of a best simultaneous approximation of two functions in abstract spaces and with respect to the l_p norm, $1 \leqslant p \leqslant \infty$, has been discussed by Phillips and Goel *et al.* in [10, 11, 15]. The paper by Holland *et al.* [12] deals with approximation of more than two functions and with respect to the supremum norm. Simultaneous approximation of one function but with several norms has been studied by Bacopoulos and his collaborators [1]–[5] and later by Dunham [9]. In particular [13, 14] appear as generalizations of [3, 9].

The notion of simultaneous approximation is based on the following.

The space $C(F, B)$ of continuous functions from a topological space F into a Banach space B over the field k, where $k = \mathbb{R}$ or $k = \mathbb{C}$ and $F \subset B$, contains the vector subspace A of all affine mappings from F into B of the form $e_{(\lambda,b)}(f) = \lambda \cdot f + b$, where $b \in B$ and $\lambda \in k$. We use the notation (λ, b) for the function $e_{(\lambda,b)}$.

Let $\|\cdot\|_E$ be a norm in A. We assume that F has more than one element and that

$$\|(0, b)\|_E = C_1 \cdot \|b\|_B \qquad \text{for some } C_1 \in R \text{ and all } b \in B. \tag{1}$$

It is easy to verify that assumption (1) on $\|\cdot\|_E$ is equivalent to the following property.

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An element $s^* \in S$, where S is a subset of B, is a best approximation in $\|\cdot\|_B$ of an element $b \in B$ by elements of S if and only if $(0, s^*)$ is a best approximation in $\|\cdot\|_E$ of $(0, b) \in A$ by elements of the subset $\{(0, s) \in A$ such that $s \in S\}$ of A.

A best $\|\cdot\|_E$-simultaneous approximation of $F \subset B$ by elements of $S \subset B$ is, by definition, an element $s^* \in S$ such that

$$\inf_{s \in S} \|(1, 0) - (0, s)\|_E = \|(1, 0) - (0, s^*)\|_E . \tag{2}$$

Examples of $\|\cdot\|_E$ *which satisfies* (1). Let $B = L_p(T, m_1)$, $1 \leqslant p < \infty$, $F \subset B$ and $V = L_q(F, m_2)$, $1 \leqslant q < \infty$. Assume that for every $e_{(\lambda,b)}$

$$\|e_{(\lambda,b)}\|^q_{E(V)} = \int_F \left(\int_T |\lambda \cdot f(t) - b(t)|^p \, dm_1(t)\right)^{q/p} dm_2(f) < \infty, \tag{3}$$

respectively,

$$\|e_{(\lambda,b)}\|^p_{E(V)} = \int_T \left(\int_F |\lambda \cdot f(t) - b(t)|^q \, dm_2(f)\right)^{p/q} dm_1(t) < \infty. \tag{4}$$

Equation (3) (respectively (4)) defines a norm in A. For every $b \in B$ $\|e_{(0,b)}\|_{E(V)} = C_1 \cdot \|b\|_B$ ($\|e_{(0,b)}\|_{E(V)}$), where $C_1 = (m_2(F))^{1/q}$.

The following construction generalizes the example of $\|\cdot\|_{E(V)}$ given by (3).

Construction. We define a natural mapping $\rho_B : C(F, B) \to C(F, \mathbb{R})$ by the formula $(\rho_B \Phi)(f) = \|\Phi(f)\|_B$. Let V be a subspace of $C(F, \mathbb{R})$ which contains the functions $\rho_B e_{(\lambda,b)}$ for every $e_{(\lambda,b)} \in A$ and the function $e(f) \equiv 1$. Let $\|\cdot\|_V$ be a norm on V.

The norm $\|\cdot\|_V$ is monotone if and only if for any two functions ϕ and ψ from V the inequality $0 \leqslant \phi(f) \leqslant \psi(f)$ for every $f \in F$ implies $\|\phi\|_V \leqslant \|\psi\|_V$. If the additional restriction $\phi \neq \psi$ implies $\|\phi\|_V < \|\psi\|_V$, then we call $\|\cdot\|_V$ strongly monotone.

It is easy to verify that for every monotone $\|\cdot\|_V$ the equation

$$\|e_{(\lambda,b)}\|_{E(V)} = \|\rho_B e_{(\lambda,b)}\|_V , \qquad e_{(\lambda,b)} \in A, \tag{5}$$

defines a norm $\|\cdot\|_{E(V)}$ in A. Moreover, for every $b \in B$ $\|e_{(0,b)}\|_{E(V)} = C_1 \cdot \|b\|_B$, where $C_1 = \|e\|_V$.

Remark 1. A direct generalization of $\|\cdot\|_{E(V)}$ allows one to construct examples of norms in $V = \operatorname{Span} \rho_B(A) \subset C(F, \mathbb{R})$ with different properties of monotonicity. Namely,

assume that $\{\|\cdot\|_{V(x)}\}_{x \in X}$, where X is a topological space, is a family of pseudonorms in V such that for every $\phi \neq 0$, $\phi \in V$, the function $\rho_\phi(x) = \|\phi\|_{V(x)}$ belongs to a vector subspace Y of $C(X, \mathbb{R})$ and is not identically zero.

It is easy to verify that for every monotone $\|\cdot\|_Y$ in Y the equation

$$\|\phi\|_V = \|\rho_\phi\|_Y \tag{6}$$

defines a norm in V.

It is easily checked that the monotonicity of $\|\cdot\|_{V(\alpha)}$ for every $\alpha \in X$ implies the monotonicity of $\|\cdot\|_V$. If, in addition, for every $\phi, \psi \in V$ such that $\phi \neq \psi$ and $0 \leqslant \phi(f) \leqslant \psi(f)$ for all $f \in F$ there exists an $\alpha \in X$ with $\|\phi\|_{V(\alpha)} < \|\psi\|_{V(\alpha)}$ and if $\|\cdot\|_Y$ is strongly monotone, then $\|\cdot\|_V$ is strongly monotone.

The main features of this paper are

(1) characterization of a best $\|\cdot\|_E$-simultaneous approximation of F by elements of a closed subspace $S \subset B$ (see Section 2);

(2) reduction of the $\|\cdot\|_E$-simultaneous approximation for some $\|\cdot\|_E$ to an approximation of some single element $c \in B$ (see Section 3);

(3) existence (see Section 4) and uniqueness (see Section 5) of a best $\|\cdot\|_E$-simultaneous approximation under some very general assumptions;

(4) exhibition of circumstances under which nonuniqueness of a best $\|\cdot\|_{E(V)}$-simultaneous approximation takes place, provided B is a strictly convex Banach space and $\|\cdot\|_V$ is a strongly monotone norm (see Section 5);

(5) an example showing that uniqueness in Theorems 2.1 and 2.2 in [11] does not take place in the case of $V = l_1$ (see Section 5).

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2. Characterization of a Best $\|\cdot\|_E$-Simultaneous Approximation of F by Elements of a Closed Subspace S of B

We shall make use of the following lemma.

LEMMA 2.1. (a) *A is a complete topological space in the norm $\|\cdot\|_E$;*

(b) *For every $\mu \in k$ and $v \in B^*$ the formula $\Phi_{(\mu,v)}(e_{(\lambda,b)}) = \mu \cdot \lambda + v(b)$, where $e_{(\lambda,b)} \in A$, defines a bounded linear functional on A,*

$$\|\Phi_{(\mu,v)}\|_{A^*} \leqslant |\mu| \cdot \frac{1}{C_2} + \|v\|_{B^*} \cdot \frac{C_2 + C_3}{C_1 \cdot C_2}, \tag{7}$$

where $C_2 = \inf_{b \in B} \|(1, b)\|_E$ and $C_3 = \|(1, 0)\|_E$.

Proof. Note first that for $\lambda \neq 0$

$$\|(\lambda, b)\|_E = |\lambda| \cdot \|(1, b/\lambda)\|_E \geq |\lambda| \cdot C_2 . \tag{8}$$

Also

$$C_1 \cdot \| b \|_B = \|(0, b)\|_E \leq \|(\lambda, b)\|_E + |\lambda| \cdot C_3 . \tag{9}$$

Assume $C_2 > 0$. Let us choose a Cauchy sequence $\{(\lambda_n , b_n)\}_{n\geq 1}$ in A. From inequalities (8) and (9) it follows that $\{\lambda_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are Cauchy sequences in k and B, respectively. Let $\lambda = \lim_{n\to\infty} \lambda_n \in k$ and $b = \lim_{n\to\infty} b_n \in B$. Since

$$\|(\lambda, b) - (\lambda_n , b_n)\|_E \leq C_3 \cdot |\lambda - \lambda_n| + C_1 \cdot \| b - b_n \|_B , \tag{10}$$

$\lim_{n\to\infty} \|(\lambda, b) - (\lambda_n , b_n)\|_E = 0$. Hence, the completion of A by $\|\cdot\|_E$ coincides with A. It remains to prove $C_2 > 0$.

Suppose $C_2 = 0$. For every $\epsilon > 0$ the inequality $\|(1, -b_j)\|_E \leq \epsilon$, $j = 1, 2$, implies $\| b_1 - b_2 \|_B \leq (2 \cdot \epsilon/C_1)$. Using the completness of B we obtain

$$\bigcap_{\epsilon>0} \{b \in B \text{ such that } \|(1, -b)\|_E \leq \epsilon\} \neq \emptyset . \tag{11}$$

Therefore there exists an $s^* \in B$ such that $\|(1, -s^*)\|_E = 0$. Hence, $f - s^* = 0$ for all $f \in F$ and, consequently, F is the one point set $\{s^*\}$, which contradicts our assumption that the cardinality of F is greater than 1. This proves $C_2 > 0$ and statement (a).

To prove inequality (7) let us mention that (8) implies

$$\| \Phi_{(\mu,0)}\|_{A^*} = |\mu| \cdot (1/C_2) \tag{12}$$

and (9) implies

$$\operatorname*{Sup}_{b\in B} \frac{\| b \|_B}{\|(1, b)\|_E} \leq \frac{1}{C_1} + \frac{C_3}{C_1} \cdot \operatorname*{Sup}_{b\in B} \frac{1}{\|(1, b)\|_E} = \frac{C_2 + C_3}{C_1 \cdot C_2} . \tag{13}$$

Therefore

$$\| \Phi_{(0,v)} \|_{A^*} = \operatorname*{Sup}_{b\in B} \frac{| v(b)|}{\|(1, b)\|_E} \leq \| v \|_{B^*} \cdot \frac{C_2 + C_3}{C_1 \cdot C_2} . \tag{14}$$

Combining estimates (12) and (14) we obtain (7).

The main result of this section is

THEOREM 1. *Let S be a closed subspace of a Banach space B. Then an element $s^* \in S$ is a best $\|\cdot\|_E$-simultaneous approximation of F by S if and only if there exists a functional $v \in B^*$ such that $v(s) = 0$ for every $s \in S$*

and $\sup_{b\in B} |\mu + v(b)|/\|(1, b)\|_E = 1$, *where* $\mu = \|(1, -s^*)\|_E$. *Moreover,*

$$C_1 \geqslant \|v\|_{B^*} \geqslant \left(1 - \frac{|\mu|}{C_2}\right) \cdot \frac{C_1 \cdot C_2}{C_2 + C_3}. \tag{15}$$

Remark 2. When B is a strictly convex Banach space, S is a finite-dimensional subspace of B, $F = \{f_1; f_2\} \subset B$ and $\|\cdot\|_E = \|\cdot\|_{E(l_\infty)}$. Theorem 1 is a strengthening of Theorem 3.2 in [10].

Proof of Theorem 1. As an immediate consequence of Lemma 2.1 and the well-known characterization of a best approximation of a point by a closed subspace S of B (see page 18 in [16], Theorem 1.1) we obtain a functional $\Phi_{(\mu,v)} \in A^*$ such that

(1) $\Phi_{(\mu,v)}(e_{(0,s)}) = 0$ for every $s \in S$;

(2) $\|\Phi_{(\mu,v)}\|_{A^*} = 1$;

(3) $\Phi_{(\mu,v)}(e_{(1,0)} - e_{(0,s^*)}) = \|(1, 0) - (0, s^*)\|_E$.

Therefore $v(s) = 0$ for every $s \in S$, $\|\Phi_{(\mu,v)}\|_{A^*} = 1$ and $\mu = \mu - v(s^*) = \Phi_{(\mu,v)}(e_{(1,0)} - e_{(0,s^*)}) = \|(1, -s^*)\|_E$. To prove $\sup_{b\in B} |\mu + v(b)|/\|(1, b)\|_E = \|\Phi_{(\mu,v)}\|_{A^*}$ it remains to show that $C_1 \geqslant \|v\|_{B^*}$.

To prove $C_1 \geqslant \|v\|_{B^*}$ we choose for every $\epsilon > 0$ an element $d_\epsilon \in B$ such that $|v(d_\epsilon) - \|v\|_{B^*}| < \epsilon$ and $\|d_\epsilon\|_B = 1$. For every $\lambda \neq 0$ $||\mu + v(d_\epsilon/\lambda)| - |\mu + (1/\lambda)\|v\|_{B^*}|| \leqslant \epsilon/|\lambda|$. The condition $\|\Phi_{(\mu,v)}\|_{A^*} = 1$ implies

$$1 \geqslant \frac{|\mu + v(d_\epsilon/\lambda)|}{\|(1, d_\epsilon/\lambda)\|_E} \geqslant \frac{|\mu + (1/\lambda)\|v\|_{B^*}| - (\epsilon/|\lambda|)}{\|(1, d_\epsilon/\lambda)\|_E}$$

$$\geqslant \frac{|\mu\cdot\lambda + \|v\|_{B^*}| - \epsilon}{\|(\lambda, d_\epsilon)\|_E}. \tag{16}$$

Therefore in (16), letting λ converge to zero and using (10), we obtain $1 \geqslant (1/C_1)\|v\|_{B^*} - (\epsilon/C_1)$, and consequently $C_1 \geqslant \|v\|_{B^*}$.

The remaining part of inequality (15) follows directly from inequality (7) putting $\|\Phi_{(\mu,v)}\|_{A^*} = 1$.

3. Reduction of $\|\cdot\|_E$-Simultaneous Approximation for Some $\|\cdot\|_E$ to an Approximation of Some Single Element $c \in B$

The main result of this section is

THEOREM 2. *Let s^* be a best approximation of some single element $c \in B$ by elements of an $S \subset B$. An element s^* is a best $\|\cdot\|_E$-simultaneous approximation of F by elements of S for every $S \subset B$ if and only if*

$$\|(\lambda, b)\|_E = \|(|\lambda|, \|b + \lambda\cdot c\|_B)\|_P \tag{17}$$

for all $(\lambda, b) \in A$ and a norm $\|\cdot\|_P$ in $\mathbb{R}^2$, which is strictly monotone as a function of the second coordinate.

Remark 3. It is easy to verify that an element c (the center $c(F)$ of F) in Theorem 2 is uniquely determined by F and $\|\cdot\|_E$ in A.

Remark 4. Assume that $F = \{f_1; f_2\}$, $\|\cdot\|_E$ in $B \oplus B$ $(A \subset B \oplus B)$ satisfies $\|(f_1, f_2)\|_E = \|(f_2, f_1)\|_E$ for all $(f_1, f_2) \in B \oplus B$, the center $c(F)$ exists and that an element s^* of a best $\|\cdot\|_E$-simultaneous approximation of F by elements of $S = B$ is unique. Then $c(F) = \frac{1}{2}(f_1 + f_2)$. Indeed, an element $c(F)$ is uniquely determined by

$$\|(1, -c(F))\|_E = \inf_{s \in B} \|(1, s)\|_E . \tag{18}$$

Also, for $c = \frac{1}{2}(f_1 + f_2)$

$$\begin{aligned}
\|(1, -c)\|_E &= \|(\tfrac{1}{2}(f_1 - f_2), \tfrac{1}{2}(f_2 - f_1))\|_E \\
&\leq \tfrac{1}{2}(\|(\tfrac{1}{2}(f_1 - f_2) - s, \tfrac{1}{2}(f_2 - f_1) - s)\|_E \\
&\quad + \|(-1) \cdot (\tfrac{1}{2}(f_1 - f_2) - s, \tfrac{1}{2}(f_2 - f_1) - s)\|_E) \\
&= \|(\tfrac{1}{2}(f_1 - f_2) - s, \tfrac{1}{2}(f_2 - f_1) - s)\|_E \\
&= \|(1, -c + s)\|_E .
\end{aligned} \tag{19}$$

Therefore $c(F) = c$. This problem has been studied by Phillips and Sahney in [15] in the case of a Hilbert space B and $\|\cdot\|_{E(l_2)}$-simultaneous approximation.

Proof of Theorem 2. The "if" part is easy. Assume $c(F) \in B$ is the center of $F \subset B$. Then $\|(1, -s)\|_E$, $s \in B$, is a strictly monotone function ψ of $\|s - c(F)\|_B$; i.e., $\|(1, -s)\|_E = \psi(\|s - c(F)\|_B)$. For every $r > 0$ and $a \geq 0$ we denote $r \cdot \psi(a/r)$ by $\hat{\psi}(r, a)$. Then, for every $\lambda \neq 0$ and $d \in B$

$$\|\lambda \cdot (1, -c(F)) + (0, d)\|_E = |\lambda| \cdot \|(1, -cF) + (d/\lambda))\|_E = \hat{\psi}(|\lambda|, \|d\|_B). \tag{20}$$

Therefore $\hat{\psi}(|\lambda|, \|\mu\|) = \|(|\lambda|, |\mu|)\|_P$, where $\|\cdot\|_P$ is a norm in $\mathbb{R}^2$. Hence

$$\begin{aligned}
\|(\lambda, b)\|_E &= \|\lambda \cdot (1, -c(F)) + (0, b + \lambda \cdot c(F))\|_E \\
&= \|(|\lambda|, \|b + \lambda \cdot c(F)\|_B)\|_P .
\end{aligned} \tag{21}$$

It is easy to check that the assumptions of Theorem 2 imply the strict monotonicity of $\|\cdot\|_P$ as a function of the second coordinate.

EXAMPLE 1. Assume B is a Hilbert space, $F \subset B$, m is a measure on F with $m(F) < \infty$, and $\|\cdot\|_V = \|\cdot\|_{L_2(F,m)}$. Then A is a Hilbert space in $\|\cdot\|_{E(V)}$ and the equality

$$\|(1, -s)\|^2_{E(V)} = m(F) \cdot \|c - s\|^2_B + \|(1, -c)\|^2_{E(V)} . \tag{22}$$

where $c = (1/m(F)) \cdot \int_F f\, dm(f)$, holds. Therefore $c(F) = c$ and $\|\cdot\|_P$ in $\mathbb{R}^2$ is given by

$$\|(\lambda, \mu)\|_P^2 = m(F) \cdot |\mu|^2 + \|(1, -c)\|_{E(V)}^2 \cdot |\lambda|^2. \tag{23}$$

In the case of $F = \{f_1; f_2\}$ and $m(f_1) = m(f_2)$ Eq. (22) is a main tool in [15] (see Remark 4).

EXAMPLE 2. Let $B = L_1(T, m)$ and $F \subset B$. Assume that for every $(\lambda, b) \in A$

$$\|(\lambda, b)\|_{E(l_\infty)} = \int_T \sup_{f \in F} |\lambda \cdot f(t) - b(t)|\, dm(t) < \infty. \tag{24}$$

Equation (24) defines a norm $\|\cdot\|_{E(l_\infty)}$ in A. It follows from $\|(1, 0)\|_{E(l_\infty)} = \int_T \sup_{f \in F} |f(t)|\, dm(t) < \infty$ that the functions $f_-^*(t) = \inf_{f \in F} f(t)$ and $f_+^*(t) = \sup_{f \in F} f(t)$ belong to $L_1(T, m)$. We denote $\frac{1}{2}(f_-^* + f_+^*) \in B$ by c. Using the identity $\max\{|a_1|; |a_2|\} = \frac{1}{2}|a_1 + a_2| + \frac{1}{2}|a_1 - a_2|$, where a_1 and $a_2 \in \mathbb{R}$, we obtain

$$\begin{aligned} \|(1, -s)\|_{E(l_\infty)} &= \int_T \max\{|f_-^*(t) - s(t)|; |f_+^*(t) - s(t)|\}\, dm(t) \\ &= \|c - s\|_B + \tfrac{1}{2}\|f_-^* - f_+^*\|_B. \end{aligned} \tag{25}$$

Therefore $c(F) = c$ and $\|\cdot\|_P$ in $\mathbb{R}^2$ is given by

$$\|(\lambda, \mu)\|_P = |\mu| + \tfrac{1}{2}\|f_-^* - f_+^*\|_B \cdot |\lambda|. \tag{26}$$

EXAMPLE 3. Let $B = L_1(T, m)$ and $F = \{f_j\}_{0 \leqslant j \leqslant N-1} \subset B$. For every $a = (a_j)_{0 \leqslant j \leqslant N-1} \in \mathbb{R}^N$ we define $a^* = (a_j^*)_{0 \leqslant j \leqslant N-1} \in \mathbb{R}^N$ by

$$a_j^* = a_{q(j)} \qquad \text{and} \qquad a_j^* \leqslant a_q^* \quad \text{for every } 0 \leqslant j \leqslant q \leqslant N - 1. \tag{27}$$

By $c \in B$ we mean the function $f^*_{\frac{1}{2}(N-1)}(t)$ for odd N and a function $c \in B$ such that $f^*_{\frac{1}{2}N-1}(t) \leqslant c(t) \leqslant f^*_{\frac{1}{2}N}(t)$ for even N. Assume that $s \in B$ satisfies $f^*_{\frac{1}{2}(N-1)-1}(t) \leqslant s(t) \leqslant f^*_{\frac{1}{2}(N-1)+1}(t)$ for odd N and $f^*_{\frac{1}{2}N-1}(t) \leqslant s(t) \leqslant f^*_{\frac{1}{2}N}(t)$ for even N. Equations

$$\begin{aligned} \|(1, -s)\|_{E(l_1)} &= \|(1, -c)\|_{E(l_1)} + \|c - s\|_B \qquad &\text{for odd } N, \\ \|(1, -s)\|_{E(l_1)} &= \|(1, -c)\|_{E(l_1)} &\text{for even } N. \end{aligned} \tag{28}$$

hold. Therefore for $\lambda \neq 0$ and $b = (1/\lambda) \cdot s$ Eq. (17) holds, where $\|\cdot\|_P$ in $\mathbb{R}^2$ is given by

$$\begin{aligned} \|(\lambda, \mu)\|_P &= |\mu| + \|(1, -c)\|_{E(l_1)} \cdot |\lambda| \qquad &\text{for odd } N, \\ \|(\lambda, \mu)\|_P &= \|(1, -c)\|_{E(l_1)} \cdot |\lambda| &\text{for even } N. \end{aligned} \tag{29}$$

However it is easy to verify that Eqs. (28) and (29) do not hold for all $s \in B$. Hence the center $c(F)$ of F does not exist.

EXAMPLE 4. Let $B = L_p(T, m)$, $F = \{f_j\}_{0 \le j \le N-1} \subset B$ and $V = l_p$, $1 \le p \le \infty$. We assume N to be odd for $p = 1$. By $\mathfrak{A}(B)$ we mean the set of all subsets $S \subset B$ such that $S = \{s \in B \text{ such that } s(t) \in D_t\}$, where $\{D_t\}_{t \in T}$ is a family of subsets of $\mathbb{R}$. The following statement allows one to reduce $\|\cdot\|_{E(l_p)}$-simultaneous approximation of F by elements of $S \in \mathfrak{A}(L_p(T, m))$ to an approximation of some single element $c_p(F) \in L_p(T, m)$.

PROPOSITION 3.1. *There exists an element $c_p(F) \in L_p(T, m)$, $1 \le p \le \infty$ such that an element $s^* \in L_p(T, m)$ is a best $\|\cdot\|_{E(l_p)}$-simultaneous approximation of F by elements of an arbitrary $S \in \mathfrak{A}(L_p(T, m))$ if and only if s^* is a best approximation of $c_p(F)$ by elements of S.*

In the proof of Proposition 3.1 we shall make use of the following

LEMMA 3.1. *For every $a = (a_j)_{0 \le j \le N-1} \in \mathbb{R}^N$ and $s \in \mathbb{R}$ let $\Phi_p(a, s) = \sum_{j=0}^{N-1} |a_j - s|^p$, $1 \le p < \infty$, $\Phi_\infty(a, s) = \max_{0 \le j \le N-1} |a_j - s|$. The equation*

$$\Phi_p(a, s_p(a)) = \inf_{s \in \mathbb{R}} \Phi_p(a, s), \quad 1 \le p \le \infty, \tag{30}$$

has a unique solution $s_p(a)$ continuously depending on $a \in \mathbb{R}^N$ and the function $\Phi_p(a, s)$ is a strictly monotone function of $|s - s_p(a)|$.

Proof. Assume $1 \le p < \infty$ and $a_{q-1}^* < s < a_q^*$. Then

$$((d/ds)\,\Phi_p)(a, s) = p \cdot \left(\sum_{j=0}^{q-1} |a_j^* - s|^{p-1} - \sum_{j=q}^{N-1} |a_j^* - s|^{p-1} \right). \tag{31}$$

In particular, $((d/ds)\,\Phi_1)(a, s) = (2q - N)$. Hence $s_1(a) = a^*_{\frac{1}{2}(N-1)}$ (N is odd) and $\Phi_1(a, s)$ is a strictly monotone function of $|s - s_1(a)|$.

For $1 < p < \infty$ $\Phi_p(a, s)$ as a function of s belongs to $C^2(\mathbb{R} \setminus \bigcup_{j=0}^{N-1} a_j)$ and $((d^2/ds^2)\,\Phi_p)(a, s) = p \cdot (p - 1) \cdot \Phi_{p-2}(a, s) > 0$. Hence Eq. (30) is equivalent to $((d/ds)\,\Phi_p)(a, s_p(a)) = 0$ for $1 < p < \infty$ and the lemma is proved for $1 \le p < \infty$. For $p = \infty$ the lemma follows from the equality

$$\max\{|a_0^* - s|, |a_{N-1}^* - s|\} = |\tfrac{1}{2}(a_0^* + a_{N-1}^*) - s| + \tfrac{1}{2}|a_0^* - a_{N-1}^*|.$$

Proof of Proposition 3.1. Let $(c_p(F))(t) = s_p((f_j^*(t))_{0 \le j \le N-1})$. The functions $f_j^*(t)$ belong to $L_p(T, m)$, $c_p(F)$ is measurable and $f_0^*(t) \le (c_p(F))(t) \le f_{N-1}^*(t)$ for $1 \le p \le \infty$. Hence $c_p(F) \in L_p(T, m)$.

For every s_1 and $s_2 \in L_p(T, m)$ we put $s_3(t) = s_1(t)$ in case $|s_1(t) - c_p(F)(t)| < |s_2(t) - c_p(F)(t)|$ and we put $s_3(t) = s_2(t)$ for other $t \in T$. Assume $s_3 \ne s_1$ and

$s_3 \neq s_2$. Then, using the strict monotonicity of $\Phi_p(a, s)$ as a function of $\| s - s_p(a) \|$, we obtain

$$\begin{aligned} \| s_3 - c_p(F) \|_{L_p(T,m)} &< \min_{j=1,2} \| s_j - c_p(F) \|_{L_p(T,m)}, \\ \|(1, 0) - (0, s_3)\|_{E(l_p)} &< \min_{j=1,2} \|(1, 0) - (0, s_j)\|_{E(l_p)}. \end{aligned} \tag{32}$$

Proposition 3.1 follows immediately from inequalities (32).

Remark 5. Let us mention that $c_1(F) = f^*_{\frac{1}{2}(N-1)}$ and $c_\infty(F) = \frac{1}{2}(f_0^* - f^*_{N-1})$.

4. Existence of a Best $\| \cdot \|_E$-Simultaneous Approximation

The following notions are important for the problem of existence of a best $\| \cdot \|_E$-simultaneous approximation.

The weak topology induced on a Banach space B by a set W of continuous linear functionals on B we call the W-topology.

We call a subset S of B locally W-compact if and only if the intersection of S with every ball in B is W-compact.

We call a subset S of B W-nice if and only if the intersection of every closed convex subset of B with S is closed in the W-topology.

We use the notation SpanS for the closure of the linear hull of $S \subset B$. By id: SpanS $\to B$ we mean the identity operator.

Examples of Locally W-Compact and W-Nice Subsets of B. Every closed convex subset of a Banach space B is closed in the W-topology with $W = B^*$, because every convex closed set $D \subset B$ and point $b \in B \setminus D$ are separable by a bounded linear functional $v \in B^*$ [17, p. 58]. If X is a Banach space, then $B = X^*$ is locally W-compact with $W = X \subset X^{**} = B^*$ [17, p. 66]. Therefore in every reflexive Banach space B every ball is B^*-compact. Also, every compact subset in B is W-compact. Hence, a set S is locally W-compact and W-nice if:

(1) S is closed in the W-topology, id$^*(W) =$ (SpanS)* and SpanS is a reflexive Banach space (for example, SpanS is a uniformly convex Banach space). In particular SpanS $= L_p(F, m)$, where $1 < p < \infty$;

(2) S is a locally compact subset of B in the $\| \cdot \|_B$ topology and the W-topology is Hausdorff on S.

Remark 6. Let $J\colon B \to A$ be the natural embedding, $Jb = e_{(0,b)}$. Due to (1) J is an isometry modulo multiplication by a constant C_1. The mapping $J^*\colon A^* \to B^*$ is the restriction of $\Phi \in A^*$ to B and $J^*(W') \subset W$, where $W' = (J^*)^{-1}(W)$. The Hahn–Banach extension theorem implies $J^*(W') = W$.

Therefore $J: B \to J(B)$ considered as a mapping from B in the W-topology into $J(B)$ in the W'-topology is a homeomorphism.

We shall make use of the following

LEMMA 4.1. *Let S be a locally W-compact and W-nice subset of B. Then the subset $J(S)$ of A is locally W'-compact, where $W' = (J^*)^{-1}(W)$.*

Proof. Let us mention first of all that the intersection of a ball D_r in A of a radius r and center $(\lambda, d) \in A$ with $J(B)$ is a convex closed subset of $J(B)$. Using Remark 6 and the fact that S is a W-nice subset of B we obtain that $J^{-1}(D_r \cap J(B))$ is a convex closed subset of B and $J^{-1}(D_r \cap J(S)) = J^{-1}(D_r \cap J(B)) \cap S$ is a closed set in the W-topology.

In addition, the inequality

$$\begin{aligned} r &\geq \|(\lambda, d) - (0, x)\|_E \geq \|(0, x)\|_E - \|(\lambda, d)\|_E \\ &\geq C_1 \cdot \|x\|_B - |\lambda| \cdot C_3 - \|d\|_B \cdot C_1 \end{aligned} \tag{33}$$

implies that the intersection $D_r \cap J(B)$ is a bounded subset of $J(B)$ for every $r > 0$ and $(\lambda, d) \in A$.

Now the W-compactness of $J^{-1}(D_r \cap J(S))$ follows from the inclusion

$$J^{-1}(D_r \cap J(S)) \subset D_R(B) \cap S, \tag{34}$$

where $R = R(r, \lambda, d) = (r + |\lambda| \cdot C_3)/C_1 + \|d\|_B$ and $D_R(B)$ is a ball of radius R and center 0 in B, and from the closedness of $J^{-1}(D_r \cap J(S))$ in the W-topology. Hence, using Remark 6, $D_r \cap J(S)$ is W'-compact.

The main result of this section is

THEOREM 3. *Let S be a locally W-compact and W-nice subset of a Banach space B. Then for every $F \subset B$ and $\|\cdot\|_E$ satisfying* (1) *there exists a best $\|\cdot\|_E$-simultaneous approximation of F by elements of S.*

Remark 7. Theorem 3 is a generalization of Lemma 2.2 and Proposition 4.1 in [10] and Theorems 1 and 2 in [12] about the existence of a best $\|\cdot\|_{E(l_\infty)}$-simultaneous approximation in the case when F is a compact. The restrictions on B and S in [10, 12] are: B is a strictly convex Banach space and S is a finite-dimensional subspace of B, or B is a uniformly convex Banach space and S is a closed convex set.

Proof. To prove Theorem 3 we consider the intersections with $J(S)$ of balls in A of radius r and center $(1, 0)$. For every $r > \inf_{s \in S} \|(1, 0) - (0, s)\|_E = r_0$, the set $D_r \cap J(S) \neq \varnothing$ and is W'-compact (see Lemma 4.1). Therefore the intersection

$$U = \bigcap_{r > r_0} (D_r \cap J(S)) \neq \varnothing. \tag{35}$$

To complete the proof it is enough to mention that every point of $U \subset J(S)$ considered as a point of S realizes a best $\|\cdot\|_E$-simultaneous approximation of F by elements of S.

5. Uniqueness of a Best $\|\cdot\|_{E(V)}$-Simultaneous Approximation

The following example is typical for nonuniqueness of a best $\|\cdot\|_{E(V)}$-simultaneous approximation, provided B is a strictly convex Banach space, S is a convex subset of B, and $\|\cdot\|_V$ is a strongly monotone norm.

EXAMPLE. Let $V = L_1(F, m)$ and B be a strictly convex Banach space. Assume the following.

C.1. For some $\hat{b}_1 \neq \hat{b}_2$, $\hat{b}_1$ and $\hat{b}_2$ from B, $F = F_1 \cup F_2$, where $F_1 \subset l_1 = \{b \in B \mid \exists \alpha \in \mathbb{R},\ \alpha \geqslant 1 \text{ such that } b = \alpha \cdot \hat{b}_1 + (1 - \alpha) \cdot \hat{b}_2\}$, $F_2 \subset l_2 = \{b \in B \mid \exists \alpha \in \mathbb{R},\ \alpha \leqslant 0 \text{ such that } b = \alpha \cdot \hat{b}_1 + (1 - \alpha) \cdot \hat{b}_2\}$. We use the notation $l = \{b \in B \mid \exists \alpha \in \mathbb{R}, \text{ such that } b = \alpha \cdot \hat{b}_1 + (1 - \alpha) \cdot \hat{b}_2\}$ and $I = l \backslash (l_1 \cup l_2)$.

Assume $m(F_1) = m(F_2)$. Then the function $\phi(b) = \|(1, -b)\|_{E(V)}$ is constant on I.

Remark 8. The above example for $F_1 = \{f_1\}$ and $F_2 = \{f_2\}$ contradicts Theorems 2.1 and 2.2 in [11] in the case of $V = l_1$.

The following examples illustrate notions which are important for the problem of uniqueness of a best $\|\cdot\|_{E(V)}$-simultaneous approximation.

EXAMPLE OF STRONG MONOTONICITY. The $L_p(F, m)$-norm, $1 \leqslant p < \infty$, is a strongly monotone norm if and only if the support of the measure m is equal to F.

DEFINITION. The norm $\|\cdot\|_V$ is strictly monotone on V at a point $\phi \in V$ if and only if for every $\psi \in V$ the inequality $0 \leqslant \phi(f) < \psi(f)$ for all $f \in F$ implies $\|\phi\|_V < \|\psi\|_V$. $\|\cdot\|_V$ is strictly monotone if it is strictly monotone on V at every point $\phi \in V$.

EXAMPLE OF STRICT MONOTONICITY. It is easily checked that the supremum norm is a strictly monotone norm if and only if F is compact. Moreover, if F is not compact, but for every $b \in B$ there exists $f' \in F$ such that $\|f' + b\|_B = \sup_{f \in F} \|f + b\|_B$, then $\|\cdot\|_V$ is strictly monotone on V at every point $\phi \in \rho_B(A)$. If the supremum norm is strictly monotone on V at every point $\phi \in \rho_B(A)$, then the following condition holds.

C.2. For every $\phi_j \in \rho_B(A)$, $j = 1, 2, 3$, with $\|\phi_1\|_V = \|\phi_2\|_V = \|\phi_3\|_V$ and $\alpha \in \mathbb{R}$, $0 < \alpha < 1$, such that $0 \leqslant \phi_3(f) \leqslant \alpha \cdot \phi_1(f) + (1 - \alpha) \cdot \phi_2(f)$ for all $f \in F$, there exists $f' \in F$ satisfying $\phi_1(f') = \phi_2(f') = \phi_3(f')$.

Indeed, strict monotonicity of $\|\cdot\|_V$ on V at every point $\phi \in \rho_B(A)$ implies the existence of an $f' \in F$ such that

$$\begin{aligned} \|\phi_3\|_V = \sup_{f \in F} |\phi_3(f)| &= \phi_3(f') \leqslant \alpha \cdot \phi_1(f') + (1 - \alpha) \cdot \phi_2(f') \\ &\leqslant \alpha \cdot \sup_{f \in F} |\phi_1(f)| + (1 - \alpha) \cdot \sup_{f \in F} |\phi_2(f)| \\ &= \alpha \cdot \|\phi_1\|_V + (1 - \alpha) \cdot \|\phi_2\|_V . \end{aligned} \tag{36}$$

Therefore, using $\|\phi_1\|_V = \|\phi_2\|_V = \|\phi_3\|_V$, we obtain $\phi_3(f') = \alpha \cdot \phi_1(f') + (1 - \alpha) \cdot \phi_2(f')$ and, moreover, $\phi_j(f') = \|\phi_j\|_V$, $j = 1, 2, 3$; i.e., $\phi_1(f') = \phi_2(f') = \phi_3(f')$.

Condition C.2 also holds in the following two examples.

(1) $\|\cdot\|_V$ is strictly monotone and strictly convex, in particular the $L_p(F, m)$ norm, where $1 < p < \infty$.

Using the existence of $f' \in F$ such that $\phi_3(f') = \alpha\phi_1(f') + (1 - \alpha)\, \phi_2(f')$ and the equality $\phi_1 = \phi_2$ (otherwise $\alpha \|\phi_1\|_V + (1 - \alpha)\|\phi_2\|_V = \|\phi_3\|_V < \|\alpha \cdot \phi_1 + (1 - \alpha) \cdot \phi_2\|_V \leqslant \alpha\|\phi_1\|_V + (1 - \alpha)\|\phi_2\|_V$) we obtain $\phi_1(f') = \phi_2(f') = \phi_3(f')$.

(2) $\|\cdot\|_V$ is the $L_1(F, m)$ norm, the support of the measure m is equal to F and F is a connected subset of B.

The strong monotonicity of $\|\cdot\|_{L_1(F,m)}$ implies that for every $f \in F$ $\phi_3(f) = \alpha \cdot \phi_1(f) + (1 - \alpha) \cdot \phi_2(f)$. The continuity of $\phi_j(f)$, $j = 1, 2$, and the connectivity of F imply the existence of an $f' \in F$ such that $\phi_1(f') = \phi_2(f')$ (otherwise either $\phi_1(f) < \phi_2(f)$ for all $f \in F$ or $\phi_2(f) < \phi_1(f)$ for all $f \in F$. So in both cases $\|\phi_1\|_V \neq \|\phi_2\|_V$). Hence $\phi_1(f') = \phi_2(f') = \phi_3(f')$.

We shall make use of the following

LEMMA 5.1. (a) *Let B be a strictly convex Banach space and $\|\cdot\|_V$ be a strongly monotone norm. If the function $\phi(b) = \|(1, -b)\|_{E(V)}$ is not strictly convex on B, then* C.1 *holds and ϕ is strictly convex on $B \setminus I$. Moreover, ϕ is linear on I. If V is a strictly convex Banach space, then A is strictly convex.*

(b) *Let B be a strictly convex Banach space and $\|\cdot\|_V$ be a monotone norm. Assume* C.2 *holds. Then for every $r > 0$ the function $\phi(b) = \|(1, -b)\|_{E(V)}$ is strictly convex on $S_r = \{b \in B \mid \phi(b) = r\} \subset B$, i.e., $\phi(\alpha \cdot b_1 + (1 - \alpha) \cdot b_2) < \alpha \cdot \phi(b_1) + (1 - \alpha) \cdot \phi(b_2)$, $0 < \alpha < 1$, for every $b_1 \neq b_2$, b_1 and b_2 from S_r.*

Proof of Statement (a). For every $\alpha, \beta \in k$ and $b_j \in B, j = 1, 2$, the inequalities

$$\begin{aligned}\|\alpha(1, -b_1) + \beta(1, -b_2)\|_{E(V)} &= \|\rho_B(\alpha(1, -b_1) + \beta(1, -b_2))\|_V \\ &\leqslant \| |\alpha| \cdot \rho_B((1, -b_1)) + |\beta| \cdot \rho_B((1, -b_2))\|_V \\ &\leqslant |\alpha| \cdot \|(1, -b_1)\|_{E(V)} + |\beta| \cdot \|(1, -b_2)\|_{E(V)}\end{aligned} \tag{37}$$

follow from the monotonicity of $\|\cdot\|_V$.

Assume that function ϕ is not strictly convex on B; i.e., that there exists $\alpha \in \mathbb{R}$, $0 < \alpha < 1$, b_1 and b_2 from B, $b_1 \neq b_2$ such that

$$\phi(\alpha \cdot b_1 + (1 - \alpha) \cdot b_2) = \alpha \cdot \phi(b_1) + (1 - \alpha) \cdot \phi(b_2). \tag{38}$$

Then setting $\beta = 1 - \alpha$ in (37) and using the strong monotonicity of $\|\cdot\|_V$ we obtain

$$\alpha \cdot \|f - b_1\|_B + (1 - \alpha) \cdot \|f - b_2\|_B = \|f - (\alpha \cdot b_1 + (1 - \alpha) \cdot b_2\|_B \tag{39}$$

for every $f \in F$. Therefore for every $f \in F$ there exists $\mu_f \in k$ such that

$$f - b_1 = \mu_f(f - b_2) \qquad \text{or} \qquad f - b_2 = 0 \tag{40}$$

($\mu_f \neq 1$ as $b_1 \neq b_2$). Since $\alpha \cdot |\mu_f| + (1 - \alpha) = |\alpha \cdot \mu_f + (1 - \alpha)|$ (by substituting (40) into (39)), μ_f is real and positive. Putting $\epsilon_f = (1/(1 - \mu_f))$ we obtain $f = \epsilon_f \cdot b_1 + (1 - \epsilon_f) \cdot b_2$, $\epsilon_f \geqslant 1$ or $\epsilon_f \leqslant 0$. Hence C.1 holds with

$$\hat{b}_1 = b_1 \qquad \text{and} \qquad \hat{b}_2 = b_2. \tag{41}$$

Notice that the equation $\|\alpha(1, -b_1) + (1 - \alpha)(1, -b_2)\|_{E(V)} = \alpha \cdot \|(1, -b_1)\|_{E(V)} + (1-\alpha) \cdot \|(1, -b_2)\|_{E(V)}$ coincides with (38). Therefore, using the triangle inequality for $\|\cdot\|_{E(V)}$, we obtain (38) for all $0 \leqslant \alpha \leqslant 1$. Hence $\phi(b)$ is linear on I. We have deduced from (38) that the elements of F, $\hat{b}_1 = b_1$, and $\hat{b}_2 = b_2$ belong to the same real line l in B. Hence (38) holds only if $b_j \in l$, $j = 1, 2$. This proves strict convexity of $\phi(b)$ on $B \setminus l$.

Assume now that $\|\cdot\|_V$ is strongly monotone and strictly convex. Let

$$\|(\lambda_1, d_1) + (\lambda_2, d_2)\|_{E(V)} = \|(\lambda_1, d_1)\|_{E(V)} + \|(\lambda_2, d_2)\|_{E(V)}. \tag{42}$$

Using (42), the strict convexity of $\|\cdot\|_V$ and

$$\begin{aligned}\|(\lambda_1, d_1) + (\lambda_2, d_2)\|_{E(V)} &= \|\rho_B((\lambda_1, d_1) + (\lambda_2, d_2))\|_V \\ &\leqslant \|\rho_B((\lambda_1, d_1)) + \rho_B((\lambda_2, d_2))\|_V \\ &\leqslant \|(\lambda_1, d_1)\|_{E(V)} + \|(\lambda_2, d_2)\|_{E(V)}\end{aligned} \tag{43}$$

we obtain

$$\|\lambda_1 f + d_1 + \lambda_2 f + d_2\|_B = \|\lambda_1 f + d_1\|_B + \|\lambda_2 f + d_2\|_B \tag{44}$$

for every $f \in F$. Since $\|\cdot\|_B$ is strictly convex, for every $f \in F$ there exists $\mu_f \in k$ such that

$$\lambda_1 f + d_1 = \mu_f(\lambda_2 f + d_2) \qquad \text{or} \qquad \lambda_2 f + d_2 = 0. \tag{45}$$

Since $|\mu_f - 1| = |\mu_f| - 1$, μ_f is real and positive. Also, the strict convexity of $\|\cdot\|_V$ implies the existence of $\nu \in \mathbb{R}$ such that either
$\|\lambda_1 f + d_1\|_B = \nu \cdot \|\lambda_2 f + d_2\|_B$ for all $f \in F$ or $\lambda_2 f + d_2 = 0$ for all $f \in F$. (46)
Combining (45) and (46) we obtain that either (λ_1, d_1) and (λ_2, d_2) are linearly dependent or the cardinality of F is 1. The latter contradicts our assumption that F is not a one point subset of B. This proves statement (a).

Proof of Statement (b). Assume that the function ϕ is not strictly convex on S_r: i.e., there exist $\alpha \in R$, $0 < \alpha < 1$, b_1 and b_2 in B, $b_1 \neq b_2$ such that (38) takes place and $r = \phi(b_1) = \phi(b_2)$. Let $\phi_j(f) = \|f - b_j\|_B$, $j = 1, 2$, and $\phi_3(f) = \|f - (\alpha \cdot b_1 + (1 - \alpha) \cdot b_2)\|_B$. Then

$$\|\phi_3\|_V = \phi(\alpha \cdot b_1 + (1 - \alpha) \cdot b_2) = \phi(b_j) = \|\phi_j\|_V, \qquad j = 1, 2,$$
and $$\tag{47}$$
$$0 \leq \phi_3(f) \leq \alpha \cdot \phi_1(f) + (1 - \alpha) \cdot \phi_2(f).$$

Applying C.2 we find $f' \in F$ such that

$$\|f' - b_1\|_B = \|f' - b_2\|_B = \|f' - (\alpha \cdot b_1 + (1 - \alpha) \cdot b_2)\|_B, \qquad 0 < \alpha < 1. \tag{48}$$

Therefore, using strict convexity of $\|\cdot\|_B$, we obtain $\|f' - b_1\| = \|f' - b_2\|$, hence $b_1 = b_2$, which contradicts our assumption. This proves Lemma 5.1.

The main result of this section is

THEOREM 4. *Let S be a convex subset of a strictly convex Banach space B.*

(a) *If $\|\cdot\|_V$ is strongly monotone norm, then for every $F \subset B$ either the set S^* of all best $\|\cdot\|_{E(V)}$-simultaneous approximations of F by S is a one point subset of S or S^* is an interval and* C.1 *holds with $S^* = I \cup \{\hat{b}_1 ; \hat{b}_2\}$.*

(b) *If $\|\cdot\|_V$ is monotone and* C.2 *holds, then for every $F \subset B$ there exists at most one best $\|\cdot\|_{E(V)}$-simultaneous approximation of F by elements of S.*

Remark 9. Statement (b) of Theorem 4 is a generalization of Propositions 3.1 and 4.1 in [10], Theorems 1 and 2 in [12] and Theorems 2.1 and 2.2 in [11] (see Remark 8) about uniqueness of a best $\|\cdot\|_{E(V)}$-simultaneous approximation of F by elements of S in the case when F is compact and $\|\cdot\|_V$ is the supremum norm (in [12]), or F is a two point subset of B and $V = \ell_p$ (with $1 \leq p < \infty$ in [11] and $p = \infty$ in [10]). The restrictions on B and S in [10–12] are: B is a strictly convex Banach space and S is a finite-dimensional subspace of B, or B is a uniformly convex Banach space and S is a closed convex subset of B. Our assumptions on B and S allow B to be a strictly convex Banach space and S to be a convex subset of B.

Proof of Theorem 4. To prove Theorem 4 it is enough to mention that the strict convexity of the function $\phi(b) = \|(1, 0) - (0, b)\|_{E(V)}$ on $S_r = \{b \in B \mid \phi(b) = r\}$ and the convexity of S imply the existence of at most one best approximation of $(1, 0)$ by elements of the convex subset $J(S)$ of A. Hence. statement (b) follows from Lemma 5.1(b).

To prove statement (a) assume S^* contains two elements b_1 and b_2, $b_1 \neq b_2$. Then ϕ is not strictly convex (because $b = \alpha \cdot b_1 + (1 - \alpha) \cdot b_2 \in S$ for all α, $0 \leqslant \alpha \leqslant 1$, and $\phi(\alpha \cdot b_1 + (1 - \alpha) \cdot b_2) \leqslant \alpha \cdot \phi(b_1) + (1 - \alpha) \cdot \phi(b_2) = \inf_{s \in S} \phi(s) \leqslant \phi(\alpha \cdot b_1 + (1 - \alpha) \cdot b_2)$ imply Eq. (38)). Therefore (see Lemma 5.1(a)) condition C.1 holds, ϕ is strictly convex on $B \setminus I$, and ϕ is constant on I. Moreover, $\hat{b}_j = b_j \in I$, $j = 1, 2$ (see (41)). Hence $S^* \subset I$. Since S^* is convex together with S, S^* is an interval. Choosing b_1 and b_2 to be the ends of S^* we obtain $S^* = I \cup \{\hat{b}_1 ; \hat{b}_2\}$.

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